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FORMATION OF CRACKS ON COMPRESSING AN UNBOUNDED BRITTLE BODY WITH A CIRCULAR OPENING*

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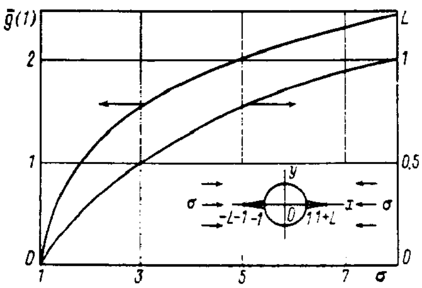
A simplified model of a brittle body /1/ is used as a basis for investigating the appearance of cracks originating at the boundary of a circular cavity in a body in a state of plane deformation caused by uniaxial compression at infinity. The singular integral equation of the problem is reduced to an integral Fredholm equation with a degenerate kernel. The solution is obtained in the form of a Fourier series in terms of Legendre polynomials.

1. Formulation of the problem. Basic relationships. Let the body occupy the outside of a unit circle whose centre coincides with the origin of coordinates Oxy and is in a state of plane deformation caused by the stresses at infinity $\sigma_x^\infty = -p\ (p > 0)$, $\sigma_y^\infty = \tau_{xy}^\infty = 0$. The elastic stresses are given by the Kolosov functions /2/

$$\Phi(z) = -\frac{p}{4}\left(1 - \frac{2}{z^2}\right), \quad \Psi(z) = \frac{p}{2}\left(1 - \frac{1}{z^2} + \frac{3}{z^4}\right)$$

Here $z = x + iy$, $|z| \geqslant 1$. The tensile stresses are largest on the opening contour at the points $z = \pm 1$ and are equal to $\sigma_y(\pm 1, 0) = p$. Maximum compressive stresses are attained at the points $z = \pm i$, $\sigma_x(0, \pm 1) = -3p$. The ratio of the compressive to the tensile strength $\varkappa$ is much greater than unity for a number of brittle materials. It was found e.g. in /3, 4/ that for rocks $\varkappa \sim 10^2$, and for glass and ceramics $\varkappa \sim 10$. We shall assume that $\varkappa > 3$ for the body in question. Then for $p > \sigma_0$ (σ_0 is the magnitude of the resistance to fracture), two fracture cracks symmetrically distributed about the Ox axis form within the body (see the figure).



According to the simplified model of a brittle body /1/ the crack surfaces pull towards each other due to the stresses σ_0, provided that the distance δ separating them does not exceed the value of the material constant δ_* (incipient cracks or the zone of weakened bonds), when $\delta > \delta_*$, the crack surfaces do not interact with each other (developed cracks). We denote by L the length of the incipient crack. Assuming that the elastic displacements $u_y(x, 0)$ undergo a jump $g(x) = u_y^+(x, 0) - u_y^-(x, 0)$ on the segment of the real axis $1 \leqslant |x| \leqslant 1 + L$ and using well-known relations /5/, we can write the singular integral equation for determining the displacement jump density

$$2D \int_1^{(1+L)^2} \left\{\frac{1}{t - r} + \frac{1}{tr} + \frac{r - t}{tr(tr - 1)^2} + \frac{4(t - 1)(r - 1)}{(tr - 1)^3}\right\} \mu(t)\, dt = \tag{1.1}$$
$$\frac{p}{2}\left(\frac{1}{r} - \frac{3}{r^2}\right) + \sigma_0, \quad D = \frac{G}{4\pi(1 - \nu)}$$

where G is shear modulus and ν is Poisson's ratio.

Approximate methods were used earlier /5, 6/ to solve equations with analogous kernels. The method used below reduces the singular integral Eq.(1.1) to an integral Fredholm equation with a degenerate kernel whose solution in the class of bounded functions is sought in the form of a Fourier series in Legendre polynomials.

2. Solution of the integral equation. We divide (1.1) by σ_0 and pass to the equation on the segment $[0, 1]$ using the linear transformation $t = \varepsilon\tau + 1$, $r = \varepsilon\xi + 1$. Having separated explicitly the terms of the kernel containing the singularities, we write ($\sigma = p/\sigma_0$ is the dimensionless stress at infinity)

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$$\int_0^1 (S+F)\,\mu_0\,d\tau = 1 + \frac{\sigma}{2}\left[\frac{1}{1+\varepsilon\xi} - \frac{3}{(1+\varepsilon\xi)^2}\right] \tag{2.1}$$

$$S = \frac{1}{\tau-\xi} + \frac{1}{\tau+\xi} + \frac{2\tau}{(\tau+\xi)^2} - \frac{4\tau^2}{(\tau+\xi)^3}$$

$$F = \varepsilon\left\{\frac{\tau-\xi}{\tau+\xi}\,\frac{1}{1+A} + \frac{\tau\xi(\tau-\xi)}{(\tau+\xi)^3}\left[\frac{1}{1+A} + \frac{1}{(1+A)^2}\right] - \right.$$
$$\left. \frac{4(\tau\xi)^2}{(\tau-\xi)^4}\left[\frac{1}{1+A} + \frac{1}{(1+A)^2} + \frac{1}{(1+A)^3}\right] + \frac{1+\varepsilon(\tau-\xi)}{(1+\varepsilon\tau)(1+\varepsilon\xi)}\right\}$$

$$A = \frac{\varepsilon\tau\xi}{\tau+\xi}, \quad \varepsilon = (1+L)^2 - 1, \quad \mu_0 = \frac{2D\mu(1+\varepsilon\tau)}{\sigma_0}$$

The function $F(\tau, \xi)$ is bounded for $0 \leqslant \tau, \xi \leqslant 1$; therefore it can be represented, with prescribed accuracy, by a segment of a Fourier series in displaced Lenendre polynomials

$$F(\tau, \xi) = \sum_{k,l} a_{kl} R_k(\tau) R_l(\xi)$$

where the coefficients a_{kl} are given by the double integral

$$a_{kl} = (2k+1)(2l+1)\int_0^1\int_0^1 F(\tau,\xi)\,R_k(\tau)\,R_l(\xi)\,d\tau\,d\xi$$

Putting

$$c_k = \int_0^1 \mu_0(\tau)\,R_k(\tau)\,d\tau \tag{2.2}$$

we reduce (2.1) to the form

$$\int_0^1 S(\tau,\xi)\,\mu_0(\tau)\,d\tau = f(\xi) - \sum_{k,l} a_{kl} c_k R_k(\xi) \tag{2.3}$$

where $f(\xi)$ is the right side of the Eq. (2.1).

Applying the Mellin transform to (2.3) we obtain a functional Wiener-Hopf equation in the strip $-1 < \operatorname{Re} s < 0$

$$\Phi^-(s) = K(s)\,G(s)\,[Q(s)/(s+1) + \Phi^+(s)] \tag{2.4}$$

$$\Phi^-(s) = \int_0^1 \mu_0(\tau)\,\tau^s\,d\tau, \quad \Phi^+(s) = \int_1^\infty \sigma_y(\tau, 0)\,\tau^s\,d\tau$$

$$K(s) = \operatorname{ctg}\frac{\pi s}{2}, \quad G(s) = \frac{1}{\Delta(s)}\sin^2\frac{\pi s}{2}$$

$$Q(s) = 1 + \sigma w(s) - (s+1)\sum_{k,l} a_{kl} M(s,l)\,c_k$$

$$w(s) = 0{,}5\,(1+\varepsilon)^{-1}\,[(1+3s)\,{}_2F_1(1, 1, s+2, \varepsilon(1+\varepsilon)^{-1}) - 3(s-1)]$$

$$M(s,l) = \Gamma^2(s+1)/[\Gamma(s+l-2)\,\Gamma(s-l+1)]$$

$$\Delta(s) = \sin^2 \pi s/2 - s^2$$

(${}_2F_1(\alpha, \beta, \gamma, z)$ is the hypergeometric Gauss function).

Using the results of /7/ in which a functional equation analogous to (2.4) was obtained for the case $F(\tau, \xi) \equiv 0$, we write the following expressions for the unknown functions $\Phi^\pm(s)$:

$$\Phi^-(s) = \frac{-2\varphi^-(s)\,K^-(s)}{G^-(s)}, \quad \Phi^+(s) = \frac{-s\varphi^+(s)}{K^+(s)\,G^+(s)} \tag{2.5}$$

$$K^\pm(s) = \frac{\Gamma(1 \mp s/2)}{\Gamma(1/2 \mp s/2)}, \quad G^\pm(s) = \exp\left[\frac{1}{2\pi i}\int_C \frac{\ln G(t)\,dt}{t-s}\right]$$

$$\varphi^\pm(s) = \frac{1}{2\pi i}\int_C \frac{\varphi(t)\,dt}{t-s}, \quad \varphi(s) = \frac{K^+(s)\,G^+(s)\,Q(s)}{s(s+1)}$$

($\Gamma(s)$ is the gamma function, and C is a straight line lying in the strip $-1 < \operatorname{Re} s < 0$). The plus and minus indices mean that the function is analytic and has no zeros in the region $\operatorname{Re} s < 0$ and $\operatorname{Re} s > -1$, respectively. The following relations hold in the strip $-1 < \operatorname{Re} s < 0$:

$$K(s) = 2K^+(s)\,K^-(s)/s, \quad G(s) = G^+(s)\,G^-(s)$$

Applying the theorem of residues to (2.5), we obtain

$$\Phi^-(s) = \frac{K^-(s)}{2G^-(s)(s+1)}\sum_j \frac{L(s_j)\,[Q(s) - Q(s_j)]}{s - s_j} \tag{2.6}$$

$$L(s_j) = \frac{t_j G^-(s_j)\sin \pi s_j}{K^-(s_j)}$$

where s_j are roots of the equation $\Delta(s) = 0$ lying in the right half-plane and t_j is the residue of the function $1/\Delta(s)$ at the point $s = s_j$. In deriving expressions (2.6) we used the condition that the stresses are bounded at the tip of the incipient crack (or a condition equivalent to $\mu_0(1) = 0$) which, according to the Abel-type theorem /8/ has the form

$$\sum_j \frac{L(s_j)\,Q(s_j)}{1+s_j} = 0 \qquad (2.7)$$

Using the inverse Mellin transform

$$\left(\mu_0(\tau) = \frac{1}{2\pi i}\int_C \Phi^-(s)\,t^{-s-1}\,ds\right)$$

we obtain

$$\mu_0(\tau) = \sum_{i,j} \frac{N(s_j)\,L(s_j)\,[Q(s_j) - Q(s_i)]}{s_i + s_j}\,(\tau^{s_j-1} - 1) \qquad (2.8)$$

$$N(s_i) = \frac{s_i t_i \sin \pi s_i}{8K^-(s_i)\,G^-(-s_i)\,(1-s_i)}$$

Expression (2.8) represents an integral Fredholm equation with degenerate kernel, since $Q(s)$ contains c_k given by the integral (2.2). We shall seek its solution in the form of a Fourier series in displaced Legendre polynomials, taking into account the expansion

$$\tau^{s-1} - 1 = \sum_k (2k+1)\,\alpha_k R_k(\tau), \quad \alpha_k = M(s-1, k)$$

We obtain the following system of linear equations for c_k :

$$\sum_{k=0}^{n} \sum_{i,j,l} E_{ijm} a_{kl} \left[(1+s_j)\,M(s_j, l) - (1-s_i)\,M(-s_i, l)\right] c_k - c_m + \sigma \sum E_{ijm}\,(w(s_j) - w(-s_i)) = 0 \qquad (2.9)$$

$$m = 0, 1, \ldots, n, \qquad E_{ijm} = N(s_i)\,L(s_j)\,M(s_i - 1, m)\,(s_i + s_j)$$

It is convenient to assume that the length L of the incipient crack is known. Then, adding condition (2.7) to (2.9) we obtain an algebraic system of $n+2$ linear equations for determining c_k, σ $(k = 0, 1, \ldots, n)$.

The figure shows the dependence of the length and development of the crack on the load σ.

When $L \sim 1$, finite increments in σ are matched by small changes in L. In the case of a real material we have a solution when $\varkappa 3 > \sigma$. This condition can be made sharper if we include the stresses caused at the contour of the opening by the incipient cracks. Using expression (2.14) or /5/, we obtain

$$|\sigma_x(0, \pm 1)| = 4 \int_1^{1+L} \frac{(t^2-1)\,|\mu(t)|}{t\,(t^2+1)^2}\,dt \leqslant L^3 (L+2) \max_t |\mu(t)|$$

For example, for $L = 1$ we have $|\sigma_x(0, \pm 1)| < 3 \cdot 10^{-2}$, i.e. the contribution of these stresses is not significant.

The development of the crack is governed by the formula

$$g(x) = \int_{1+L}^{x} \mu(t)\,dt$$

and reaches its maximum on the contour of the opening $(x = 1)$. The dependence of $\bar{g}(1) = 2D\sigma_0^{-1} g(1) \cdot 10^3$ on σ is shown in the figure. The relation $g(1) = \delta_*$ determines the critical load σ_* under which the crack begins to develop.

Thus the solution obtained holds for $\sigma < \varkappa 3$, $\sigma < \sigma_*$.

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